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THE GENERALIZED TOTAL GRAPH OF A COMMUTATIVE RING

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Let R be a commutative ring with nonzero identity and H be a nonempty proper subset of R such that $R\backslash H$ is a saturated multiplicatively closed subset of R. The *generalized total graph* of R is the (simple) graph $\mathrm{GT}_H(R)$ with all elements of R as the vertices, and two distinct vertices x and y are adjacent if and only if $x + y \in H$. In this paper, we investigate the structure of $\mathrm{GT}_H(R)$.

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1. Introduction

Let R be a commutative ring with nonzero identity, $Z(R)$ be its set of zero-divisors, $\mathrm{Nil}(R)$ be its ideal of nilpotent elements, and $U(R)$ be its group of units. We define a nonempty proper subset H of R to be a *multiplicative-prime* subset of R if the following two conditions hold: (i) $ab \in H$ for every $a \in H$ and $b \in R$; (ii) if $ab \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$. For example, H is multiplicative-prime subset of R if H is a prime ideal of R, H is a union of prime ideals of R, $H = Z(R)$, or $H = R\backslash U(R)$. In fact, it is easily seen that H is a multiplicative-prime subset of R if and only if $R\backslash H$ is a saturated multiplicatively closed subset of R. Thus H is a multiplicative-prime subset of R if and only if H is a union of prime ideals of R [19, Theorem 2]. Note that if H is a multiplicative-prime subset of R, then $\mathrm{Nil}(R) \subseteq H \subseteq R \backslash U(R)$; and if H is also an ideal of R, then H is necessarily a prime ideal of R. In particular, if $R = Z(R) \cup U(R)$ (e.g. R is finite), then $\mathrm{Nil}(R) \subseteq H \subseteq Z(R)$.

Let H be a multiplicative-prime subset of a commutative ring R. In this paper, we introduce the *generalized total graph* of R, denoted by $\mathrm{GT}_H(R)$, as the (simple) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in H$. For $A \subseteq R$, let $\mathrm{GT}_H(A)$ be the induced subgraph of $\mathrm{GT}_H(R)$ with all elements of A as the vertices. For example, $\mathrm{GT}_H(R \backslash H)$ is the induced subgraph of $\mathrm{GT}_H(R)$ with vertices $R \backslash H$. When $H = Z(R)$, we have that $\mathrm{GT}_H(R)$ is the so-called *total graph* of R as introduced in [6] and denoted there by $T(\Gamma(R))$. As to be expected, $\mathrm{GT}_H(R)$ and $T(\Gamma(R))$ share many properties; we invite the interested reader to compare this paper with [6]. However, the concept of generalized total graph, unlike the earlier concept of total graph, allows us to study graphs of integral domains. In particular, many of the illustrating examples in this paper involve integral domains.

Over the past several years, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures) and studying the interplay between ring-theoretic and graph-theoretic properties; see the recent survey articles [4, 21]. For example, as in [9], the *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \backslash \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. The total graph has been investigated in [2, 21–23, 25]; and several variants of the total graph have been studied in [1, 7, 8, 12, 14, 20]. In [7], we investigated the induced subgraphs of $T(\Gamma(R))$ with either $R \backslash \{0\}$ or $Z(R) \backslash \{0\}$ as vertices. Also, if H is the union of all the maximal ideals of R (i.e. $H = R \backslash U(R)$), then observe that $\overline{\mathrm{GT}_H(R)}$, the complement graph of $\mathrm{GT}_H(R)$, is the unit graph of R in the sense of [11, 24]. In [1], the authors considered the generalized total graph with respect to the set $H = S(I)$, where for a proper ideal I of R, $S(I)$ is the set of all elements of R that are not prime to I, i.e. $S(I) = \{a \in R \mid ra \in I \text{ for some } r \in R \backslash I\}$. Note that $S(I)$ is a multiplicative-prime subset of R, but our concept is more general since there are multiplicative-prime subsets not of this form. For example, let $R = \mathbb{Z}$. Then $S(\{0\}) = \{0\}$ and $S(n\mathbb{Z}) = \bigcup \{p\mathbb{Z} \mid p \text{ is prime and } p \mid n\}$ for $\{0\} \subsetneq n\mathbb{Z} \subsetneq \mathbb{Z}$; so in this case, an $S(I)$ is necessarily a finite union of prime ideals.

Let H be a multiplicative-prime subset of a commutative ring R. Since H is a union of prime ideals of R, the study of $\mathrm{GT}_H(R)$ breaks naturally into two cases depending on whether or not H is an (prime) ideal of R. In the second section, we handle the case when H is an ideal of R; and in the third section, we do the case when H is not an ideal of R. In the final section, we compute $\mathrm{GT}_H(R)$ for several classes of commutative rings R. For example, we consider idealizations, the $D + M$ construction and localizations.

Let G be a (simple) graph. We say that G is *connected* if there is a path between any two distinct vertices of G. At the other extreme, we say that G is *totally* *disconnected* if no two vertices of G are adjacent. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $\mathrm{diam}(G) = \sup\{d(x, y) \,|\, x$ and y are vertices of $G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles). We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,n}$ a *star graph*. We say that two (induced) subgraphs G_1 and G_2 of G are *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 (respectively, G_2) is adjacent (in G) to any vertex not in G_1 (respectively, G_2). By abuse of notation, we will sometimes write $G_1 \subseteq G_2$ when G_1 is a subgraph of G_2. A general reference for graph theory is [16].

Throughout this paper, all rings R are commutative with $1 \neq 0$, and H denotes a multiplicative-prime subset of R. For $A \subseteq R$, let $A^* = A \backslash \{0\}$. We say that R is *reduced* if $\mathrm{Nil}(R) = \{0\}$, and $\dim(R)$ will always mean Krull dimension. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{F}_q$ will denote the integers, rational numbers, integers modulo n, and the finite field with q elements, respectively. General references for ring theory are [18, 19].

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2. The Case When H is an Ideal of R

Let H be a multiplicative-prime subset of a commutative ring R. In this section, we study the case when H is an (prime) ideal of R (i.e. when H is closed under addition). The main goal of this section is a general structure theorem (Theorem 2.2) for $\mathrm{GT}_H(R\backslash H)$ when H is an ideal of R. We also determine the diameter and girth of the graphs $\mathrm{GT}_H(H)$, $\mathrm{GT}_H(R\backslash H)$, and $\mathrm{GT}_H(R)$.

Theorem 2.1. *Let H be a prime ideal of a commutative ring R. Then $\mathrm{GT}_H(H)$ is a complete (induced) subgraph of $\mathrm{GT}_H(R)$ and $\mathrm{GT}_H(H)$ is disjoint from $\mathrm{GT}_H(R\backslash H)$. In particular, $\mathrm{GT}_H(H)$ is connected and $\mathrm{GT}_H(R)$ is never connected.*

Proof. This follows directly from the definitions. $\square$

We now give the main result of this section. Since $\mathrm{GT}_H(H)$ is a complete subgraph of $\mathrm{GT}_H(R)$ and is disjoint from $\mathrm{GT}_H(R\backslash H)$, the next theorem also gives a complete description of $\mathrm{GT}_H(R)$. It also shows that non-isomorphic rings may have isomorphic graphs. We allow α and β to be infinite cardinals; if β is infinite, then $\beta - 1 = (\beta - 1)/2 = \beta$.

Theorem 2.2. *Let H be a prime ideal of a commutative ring R, and let $|H| = \alpha$ and $|R/H| = \beta$.*

(1) *If $2 \in H$, then $\mathrm{GT}_H(R\backslash H)$ is the union of $\beta - 1$ disjoint K^α's.*
(2) *If $2 \notin H$, then $\mathrm{GT}_H(R\backslash H)$ is the union of $(\beta - 1)/2$ disjoint $K^{\alpha,\alpha}$'s.*

Proof. (1) Assume that $2 \in H$, and let $x \in R \backslash H$. Then the coset $x+H$ is a complete subgraph of $\mathrm{GT}_H(R\backslash H)$ since $(x+z_1)+(x+z_2)=2x+z_1+z_2 \in H$ for all $z_1, z_2 \in H$ since $2 \in H$ and H is an ideal of R. Note that distinct cosets form disjoint subgraphs of $\mathrm{GT}_H(R\backslash H)$ since if $x+z_1$ and $y+z_2$ are adjacent for some $y \in R\backslash H$ and $z_1, z_2 \in H$, then $x+y=(x+z_1)+(y+z_2)-(z_1+z_2) \in H$, and hence $x-y=(x+y)-2y \in H$ since $2 \in H$ and H is an ideal of R. But then $x+H=y+H$, a contradiction. Thus $\mathrm{GT}_H(R\backslash H)$ is the union of $\beta-1$ disjoint (induced) subgraphs $x+H$, each of which is a K^{α}, where $\alpha=|H|=|x+H|$.

(2) Next, assume that $2 \notin H$, and let $x \in R\backslash H$. Then no two distinct elements in $x+H$ are adjacent since $(x+z_1)+(x+z_2) \in H$ for $z_1, z_2 \in H$ implies that $2x \in H$, and hence either $2 \in H$ of $x \in H$ since H is a prime ideal of R, a contradiction. Also, the two cosets $x+H$ and $-x+H$ are disjoint, and every element of $x+H$ is adjacent to every element of $-x+H$. Thus $(x+H)\cup(-x+H)$ is a complete bipartite (induced) subgraph of $\mathrm{GT}_H(R\backslash H)$. Furthermore, if $x+z_1$ is adjacent to $y+z_2$ for some $y \in R\backslash H$ and $z_1, z_2 \in H$, then $x+y \in H$ as in part (1) above, and hence $y+H=-x+H$. Thus $\mathrm{GT}_H(R\backslash H)$ is the union of $(\beta-1)/2$ disjoint (induced) subgraphs $(x+H)\cup(-x+H)$, each of which is a $K^{\alpha,\alpha}$, where $\alpha=|H|=|x+H|$. □

From the above theorem, one can easily deduce when $\mathrm{GT}_H(R\backslash H)$ is complete or connected, and one can explicitly compute its diameter and girth. We first determine when $\mathrm{GT}_H(R\backslash H)$ is either complete or connected.

Theorem 2.3. *Let H be a prime ideal of a commutative ring R.*

(1) $\mathrm{GT}_H(R\backslash H)$ *is complete if and only if either* $R/H \cong \mathbb{Z}_2$ *or* $R \cong \mathbb{Z}_3$.
(2) $\mathrm{GT}_H(R\backslash H)$ *is connected if and only if either* $R/H \cong \mathbb{Z}_2$ *or* $R/H \cong \mathbb{Z}_3$.
(3) $\mathrm{GT}_H(R\backslash H)$ (*and hence* $\mathrm{GT}_H(H)$ *and* $\mathrm{GT}_H(R)$) *is totally disconnected if and only if* $H=\{0\}$ (*thus R is an integral domain*) *and* $\mathrm{char}(R)=2$.

Proof. Let $|H|=\alpha$ and $|R/H|=\beta$.

(1) By Theorem 2.2, $\mathrm{GT}_H(R\backslash H)$ is complete if and only if $\mathrm{GT}_H(R\backslash H)$ is a single K^{α} or $K^{1,1}$. If $2 \in H$, then $\beta-1=1$. Thus $\beta=2$, and hence $R/H \cong \mathbb{Z}_2$. If $2 \notin H$, then $\alpha=1$ and $(\beta-1)/2=1$. Thus $H=\{0\}$ and $\beta=3$; so $R \cong R/H \cong \mathbb{Z}_3$.
(2) By Theorem 2.2, $\mathrm{GT}_H(R\backslash H)$ is connected if and only if $\mathrm{GT}_H(R\backslash H)$ is a single K^{α} or $K^{\alpha,\alpha}$. Thus either $\beta-1=1$ if $2 \in H$ or $(\beta-1)/2=1$ if $2 \notin H$; so either $\beta=2$ or $\beta=3$, respectively. Thus $R/H \cong \mathbb{Z}_2$ or $R/H \cong \mathbb{Z}_3$, respectively.
(3) $\mathrm{GT}_H(R\backslash H)$ is totally disconnected if and only if it is a disjoint union of K^1's. So $H=\{0\}$ by Theorem 2.2, and thus R must be an integral domain (since $H=\{0\}$ is a prime ideal of R) with $2 \in H$, i.e. $\mathrm{char}(R)=2$. □

Using Theorem 2.2, it is also easy to compute both the diameter and girth of $\mathrm{GT}_H(R\backslash H)$ when H is a prime ideal of R.

Theorem 2.4. *Let H be a prime ideal of a commutative ring R.*

(1) $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = 0, 1, 2,$ *or* ∞. *In particular*, $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) \leq 2$ *if* $\mathrm{GT}_H(R\backslash H)$ *is connected.*
(2) $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3, 4,$ *or* ∞. *In particular*, $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) \leq 4$ *if* $\mathrm{GT}_H(R\backslash H)$ *contains a cycle.*

Proof. (1) Suppose that $\mathrm{GT}_H(R\backslash H)$ is connected. Then $\mathrm{GT}_H(R\backslash H)$ is a singleton, a complete graph, or a complete bipartite graph by Theorem 2.2. Thus $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) \leq 2$.
(2) Suppose that $\mathrm{GT}_H(R\backslash H)$ contains a cycle. Since $\mathrm{GT}_H(R\backslash H)$ is a disjoint union of either complete or complete bipartite graphs by Theorem 2.2, it must contain either a 3-cycle or a 4-cycle. Thus $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) \leq 4$. □

The next theorem gives a more explicit description of the diameter and girth of $\mathrm{GT}_H(R\backslash H)$ when H is a prime ideal of R.

Theorem 2.5. *Let H be a prime ideal of a commutative ring R.*

(1) (a) $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = 0$ *if and only if* $R \cong \mathbb{Z}_2$.
(b) $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = 1$ *if and only if either* $R/H \cong \mathbb{Z}_2$ *and* $R \ncong \mathbb{Z}_2$ (*i.e.* $R/H \cong \mathbb{Z}_2$ *and* $|H| \geq 2$), *or* $R \cong \mathbb{Z}_3$.
(c) $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = 2$ *if and only if* $R/H \cong \mathbb{Z}_3$ *and* $R \ncong \mathbb{Z}_3$ (*i.e.* $R/H \cong \mathbb{Z}_3$ *and* $|H| \geq 2$).
(d) *Otherwise*, $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = \infty$.
(2) (a) $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$ *if and only if* $2 \in H$ *and* $|H| \geq 3$.
(a) $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 4$ *if and only if* $2 \notin H$ *and* $|H| \geq 2$.
(b) *Otherwise*, $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = \infty$.
(3) (a) $\mathrm{gr}(\mathrm{GT}_H(R)) = 3$ *if and only if* $|H| \geq 3$.
(b) $\mathrm{gr}(\mathrm{GT}_H(R)) = 4$ *if and only if* $2 \notin H$ *and* $|H| = 2$.
(c) *Otherwise*, $\mathrm{gr}(\mathrm{GT}_H(R)) = \infty$.

Proof. These results all follow directly from Theorems 2.1 and 2.2. □

The following examples illustrate the previous theorems.

Example 2.6. (a) Let $R = \mathbb{Z}$ and H be a prime ideal of R. Then $\mathrm{GT}_H(R\backslash H)$ is complete if and only if $H = 2\mathbb{Z}$, and $\mathrm{GT}_H(R\backslash H)$ is connected if and only if either $H = 2\mathbb{Z}$ or $H = 3\mathbb{Z}$. Moreover, $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = 1$ if and only if $H = 2\mathbb{Z}$, and $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = 2$ if and only if $H = 3\mathbb{Z}$. Let $p \geq 5$ be a prime integer and $H = p\mathbb{Z}$. Then $\mathrm{GT}_H(R\backslash H)$ is the union of $(p-1)/2$ disjoint $K^{\omega,\omega}$'s; so $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = \infty$. Finally, $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = \infty$ when $H = \{0\}$.

Also, $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = \infty$ if $H = \{0\}$, $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$ if $H = 2\mathbb{Z}$, and $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 4$ otherwise. Moreover, $\mathrm{gr}(\mathrm{GT}_{\{0\}}(R)) = \infty$ and $\mathrm{gr}(\mathrm{GT}_H(R)) = 3$ for any nonzero prime ideal H of R.

(b) Let $R = \mathbb{Z}_{pm} \times R_1 \times \cdots \times R_n$, where $m \geq 2$ is an integer, p is a positive prime integer, and $R_1, \ldots, R_n$ are commutative rings. Then $H = p\mathbb{Z}_{pm} \times R_1 \times \cdots \times R_n$ is a prime ideal of R. The graph $\mathrm{GT}_H(R \backslash H)$ is complete if and only if $p = 2$, and $\mathrm{GT}_H(R \backslash H)$ is connected if and only if $p = 2$ or $p = 3$. Moreover, $\mathrm{diam}(\mathrm{GT}_H(R \backslash H)) = 1$ if and only if $p = 2$, and $\mathrm{diam}(\mathrm{GT}_H(R \backslash H)) = 2$ if and only if $p = 3$. Assume that $p \geq 5$. Then $\mathrm{GT}_H(R \backslash H)$ is the union of $(p-1)/2$ disjoint $K^{\alpha,\alpha}$'s, where $\alpha = m|R_1| \cdots |R_n|$; so $\mathrm{diam}(\mathrm{GT}_H(R \backslash H)) = \infty$.

Also, $\mathrm{gr}(\mathrm{GT}_H(R \backslash H)) = 3$ if $p = 2$ and $\mathrm{gr}(\mathrm{GT}_H(R \backslash H)) = 4$ otherwise. Moreover, $\mathrm{gr}(\mathrm{GT}_H(R)) = 3$ for any prime p.

Many of the earlier results of this section can also easily be proved directly without recourse to Theorem 2.2. We give two such cases.

Theorem 2.7. *Let H be a prime ideal of a commutative ring R.*

(1) *Let G be an induced subgraph of $\mathrm{GT}_H(R \backslash H)$, and let x and y be distinct vertices of G that are connected by a path in G. Then there is a path of length at most two between x and y in G. In particular, if $\mathrm{GT}_H(R \backslash H)$ is connected, then $\mathrm{diam}(\mathrm{GT}_H(R \backslash H)) \leq 2$.*

(2) *Let x and y be distinct elements of $R \backslash H$ that are connected by a path in $\mathrm{GT}_H(R \backslash H)$. If $x + y \notin H$ (i.e. if x and y are not adjacent), then $x - (-x) - y$ and $x - (-y) - y$ are paths of length two between x and y in $\mathrm{GT}_H(R \backslash H)$.*

Proof. (1) It suffices to show that if x_1, x_2, x_3 and x_4 are distinct vertices of G and there is a path $x_1 - x_2 - x_3 - x_4$ from x_1 to x_4, then x_1 and x_4 are adjacent. Now $x_1 + x_2, x_2 + x_3, x_3 + x_4 \in H$ implies $x_1 + x_4 = (x_1 + x_2) - (x_2 + x_3) + (x_3 + x_4) \in H$ since H is an ideal of R. Thus x_1 and x_4 are adjacent.

(2) Suppose that $x + y \notin H$. Then there is a $z \in R \backslash H$ such that $x - z - y$ is a path of length two by part (1) above (note that necessarily $z \in R \backslash H$ since $x, y \in R \backslash H$). Thus $x + z, z + y \in H$ and hence $x - y = (x + z) - (z + y) \in H$ and $y - x = -(x - y) \in H$ since H is an ideal of R. Also, $x \neq -x$, $y \neq -x$ and $y \neq -y$ since $x - y \in H$ and $x + y \notin H$. Thus $x - (-x) - y$ and $x - (-y) - y$ are paths of length two between x and y in $\mathrm{GT}_H(R \backslash H)$. □

We have already observed in Theorem 2.1 that $\mathrm{GT}_H(H)$ is always connected and $\mathrm{GT}_H(R)$ is never connected when H is an ideal of R. The next theorem gives several new criteria for when $\mathrm{GT}_H(R \backslash H)$ is connected.

Theorem 2.8. *Let H be a prime ideal of a commutative ring R. Then the following statements are equivalent.*

(1) $\mathrm{GT}_H(R \backslash H)$ *is connected.*

(2) *Either $x + y \in H$ or $x - y \in H$ for every $x, y \in R \backslash H$.*

(3) *Either* $x + y \in H$ *or* $x + 2y \in H$ *for every* $x, y \in R\backslash H$. *In particular, either* $2x \in H$ *or* $3x \in H$ *(but not both) for every* $x \in R\backslash H$.
(4) *Either* $R/H \cong \mathbb{Z}_2$ *or* $R/H \cong \mathbb{Z}_3$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathrm{GT}_H(R\backslash H)$ is connected, and let $x, y \in R\backslash H$. If $x = y$, then $x - y \in H$. Hence assume that $x \neq y$. If $x + y \notin H$, then $x - (-y) - y$ is a path from x to y by Theorem 2.7(2), and thus $x - y \in H$.

(2) $\Rightarrow$ (3) Let $x, y \in R\backslash H$, and suppose that $x+y \notin H$. Since $(x+y)-y = x \notin H$, thus $x + 2y = (x + y) + y \in H$ by hypothesis. In particular, if $x \in R\backslash H$, then either $2x \in H$ or $3x \in H$. But $2x$ and $3x$ cannot both be in H since then $x = 3x - 2x \in H$, a contradiction.

(3) $\Rightarrow$ (1) Let $x, y \in R\backslash H$ be distinct elements of R such that $x + y \notin H$. Then $x + 2y \in H$ by hypothesis. Since H is an ideal of R, $x \in R\backslash H$, and $x + 2y \in H$, we have $2y \in R\backslash H$. Thus $3y \in H$ by hypothesis. Since $x + y \notin H$ and $3y \in H$, we have $x \neq 2y$, and hence $x - 2y - y$ is a path from x to y in $\mathrm{GT}_H(R\backslash H)$. Thus $\mathrm{GT}_H(R\backslash H)$ is connected.

(2) $\Rightarrow$ (4) Let $x \in R\backslash H$. Then either $x - 1 \in H$ or $x + 1 \in H$ by hypothesis, and thus either $x + H = 1 + H$ or $x + H = -1 + H$. If $2 \in H$, then $R/H \cong \mathbb{Z}_2$; otherwise, $R/H \cong \mathbb{Z}_3$.

(4) $\Rightarrow$ (2) This is clear. □

3. The Case When *H* is not an Ideal of *R*

In this section, we consider the remaining case when the multiplicative-prime subset H is not an ideal of R. Since H is always closed under multiplication by elements of R, this just means that $0 \in H$ and there are distinct $x, y \in H^*$ such that $x + y \in R\backslash H$. In this case, $\mathrm{GT}_H(H)$ is always connected (but never complete), $\mathrm{GT}_H(H)$ and $\mathrm{GT}_H(R\backslash H)$ are never disjoint subgraphs of $\mathrm{GT}_H(R)$, and $|H| \geq 3$. We first show that $\mathrm{GT}_H(R)$ is connected when $\mathrm{GT}_H(R\backslash H)$ is connected. However, we give an example to show that the converse fails.

Theorem 3.1. *Let* R *be a commutative ring and* H *be a multiplicative-prime subset of* R *that is not an ideal of* R.

(1) $\mathrm{GT}_H(H)$ *is connected with* $\mathrm{diam}(\mathrm{GT}_H(H)) = 2$.
(2) *Some vertex of* $\mathrm{GT}_H(H)$ *is adjacent to a vertex of* $\mathrm{GT}_H(R\backslash H)$. *In particular, the subgraphs* $\mathrm{GT}_H(H)$ *and* $\mathrm{GT}_H(R\backslash H)$ *of* $\mathrm{GT}_H(R)$ *are not disjoint.*
(3) *If* $\mathrm{GT}_H(R\backslash H)$ *is connected, then* $\mathrm{GT}_H(R)$ *is connected.*

Proof. (1) Every $x \in H^*$ is adjacent to 0. Thus $x - 0 - y$ is a path in $\mathrm{GT}_H(H)$ of length two between any two distinct $x, y \in H^*$. Moreover, there are nonadjacent $x, y \in H^*$ since H is not an ideal of R; so $\mathrm{diam}(\mathrm{GT}_H(H)) = 2$.

(2) Since H is not an ideal of R, there are distinct $x, y \in H^*$ such that $x + y \in R \backslash H$. Then $-x \in H$ and $x + y \in R \backslash H$ are adjacent vertices in $\mathrm{GT}_H(R)$ since $-x + (x + y) = y \in H$. The "in particular" statement is clear.

(3) Suppose that $\mathrm{GT}_H(R \backslash H)$ is connected. Since $\mathrm{GT}_H(H)$ is also connected by part (1) above, it is sufficient to show that there is a path from x to y in $\mathrm{GT}_H(R)$ for every $x \in H$ and $y \in R \backslash H$. By part (2) above, there are adjacent vertices z and w in $\mathrm{GT}_H(H)$ and $\mathrm{GT}_H(R \backslash H)$, respectively. Since $\mathrm{GT}_H(H)$ is connected, there is a path from x to z in $\mathrm{GT}_H(H)$; and since $\mathrm{GT}_H(R \backslash H)$ is connected, there is a path from w to y in $\mathrm{GT}_H(R \backslash H)$. As z and w are adjacent in $\mathrm{GT}_H(R)$, there is a path from x to y in $\mathrm{GT}_H(R)$. Thus $\mathrm{GT}_H(R)$ is connected. □

Next, we determine when $\mathrm{GT}_H(R)$ is connected and compute $\mathrm{diam}(\mathrm{GT}_H(R))$. In particular, $\mathrm{GT}_H(R)$ is connected if and only if $\mathrm{diam}(\mathrm{GT}_H(R)) < \infty$. As usual, if $A \subseteq R$, then (A) denotes the ideal of R generated by A.

Theorem 3.2. *Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R. Then $\mathrm{GT}_H(R)$ is connected if and only if $(H) = R$ (i.e. $R = (z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in H$). In particular, if H is not an ideal of R and either $\dim(R) = 0$ (e.g. R is finite) or R is an integral domain with $\dim(R) = 1$, then $\mathrm{GT}_H(R)$ is connected.*

Proof. Suppose that $\mathrm{GT}_H(R)$ is connected. Then there is a path $0 - b_1 - \cdots - b_n - 1$ from 0 to 1 in $\mathrm{GT}_H(R)$. Thus $b_1, b_1 + b_2, \ldots, b_{n-1} + b_n, b_n + 1 \in H$. Hence $1 \in (b_1, b_1 + b_2, \ldots, b_{n-1} + b_n, b_n + 1) \subseteq (H)$; so $R = (H)$.

Conversely, suppose that $(H) = R$. We first show that there is a path from 0 to x in $\mathrm{GT}_H(R)$ for every $0 \neq x \in R$. By hypothesis, $x = a_1 + \cdots + a_n$ for some $a_1, \ldots, a_n \in H$. Let $b_0 = 0$ and $b_k = (-1)^{n+k}(a_1 + \cdots + a_k)$ for every integer k with $1 \leq k \leq n$. Then $b_k + b_{k+1} = (-1)^{n+k+1} a_{k+1} \in H$ for every integer k with $0 \leq k \leq n-1$, and thus $0 - b_1 - \cdots - b_{n-1} - b_n = x$ is a path from 0 to x in $\mathrm{GT}_H(R)$ of length at most n. Now, let $0 \neq z, w \in R$. Then by the preceding argument, there are paths from z to 0 and 0 to w in $\mathrm{GT}_H(R)$. Hence there is a path from z to w in $\mathrm{GT}_H(R)$; so $\mathrm{GT}_H(R)$ is connected.

For the "in particular" statement, assume that either $\dim(R) = 0$ or R is an integral domain with $\dim(R) = 1$. Since H is a union of prime ideals of R and H is not an ideal of R, there are distinct (nonzero) prime ideals P and Q of R with $P, Q \subseteq H$. The ideals P, Q are necessarily maximal ideals of R. Thus $R = P + Q$; so $R = (p, q)$ for some $p \in P$ and $q \in Q$. Hence $\mathrm{GT}_H(R)$ is connected. □

Corollary 3.3. *Let R be a commutative ring and H be a multiplicative-prime subset of R. Then $\mathrm{GT}_H(R)$ is connected if and only if $(H) = R$.*

Proof. This follows directly from Theorems 2.1 and 3.2. □

Theorem 3.4. *Let R be a commutative ring and H a multiplicative-prime subset of R that is not an ideal of R such that $(H) = R$ (i.e.* $\mathrm{GT}_H(R)$ *is connected). Let $n \geq 2$ be the least integer such that $R = (z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in H$. Then* $\mathrm{diam}(\mathrm{GT}_H(R)) = n$. *In particular, if H is not an ideal of R and either* $\dim(R) = 0$ *(e.g. R is finite) or R is an integral domain with* $\dim(R) = 1$, *then* $\mathrm{diam}(\mathrm{GT}_H(R)) = 2$.

Proof. We first show that any path from 0 to 1 in $\mathrm{GT}_H(R)$ has length at least n. Suppose that $0 - b_1 - b_2 - \cdots - b_{m-1} - 1$ is a path from 0 to 1 in $\mathrm{GT}_H(R)$ of length m. Thus $b_1, b_1 + b_2, \ldots, b_{m-2} + b_{m-1}, b_{m-1} + 1 \in H$, and hence $1 \in (b_1, b_1 + b_2, \ldots, b_{m-2} + b_{m-1}, b_{m-1} + 1) \subseteq (H)$. Thus $m \geq n$.

Now, let x and y be distinct elements in R. We show that there is a path from x to y in $\mathrm{GT}_H(R)$ with length at most n. Let $1 = z_1 + \cdots + z_n$ for some $z_1, \ldots, z_n \in H$, and let $z = y + (-1)^{n+1}x$. Define $d_0 = x$ and $d_k = (-1)^{n+k}z(z_1 + \cdots + z_k) + (-1)^k x$ for every integer k with $1 \leq k \leq n$. Then $d_k + d_{k+1} = (-1)^{n+k+1}zz_{k+1} \in H$ for every integer k with $0 \leq k \leq n-1$ and $d_n = z + (-1)^n x = y$. Thus $x - d_1 - \cdots - d_{n-1} - y$ is a path from x to y in $\mathrm{GT}_H(R)$ with length at most n. In particular, a shortest path between 0 and 1 in $\mathrm{GT}_H(R)$ has length n, and thus $\mathrm{diam}(\mathrm{GT}_H(R)) = n$.

The "in particular" statement is clear by the proof of the last part of Theorem 3.2. □

Corollary 3.5. *Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R such that* $\mathrm{GT}_H(R)$ *is connected.*

(1) $\mathrm{diam}(\mathrm{GT}_H(R)) = d(0, 1)$.
(2) *If* $\mathrm{diam}(\mathrm{GT}_H(R)) = n$, *then* $\mathrm{diam}(\mathrm{GT}_H(R \backslash H)) \geq n - 2$.

Proof. (1) This is clear from the proof of Theorem 3.4.

(2) Since $n = \mathrm{diam}(\mathrm{GT}_H(R)) = d(0, 1)$ by part (1) above, let $0 - s_1 - \cdots - s_{n-1} - 1$ be a shortest path from 0 to 1 in $\mathrm{GT}_H(R)$. Clearly $s_1 \in H$. If $s_i \in H$ for some integer i with $2 \leq i \leq n - 1$, then the path $0 - s_i - \cdots - s_{n-1} - 1$ from 0 to 1 has length less than n, a contradiction. Thus $s_i \in R \backslash H$ for every integer i with $2 \leq i \leq n - 1$. Hence $s_2 - \cdots - s_{n-1} - 1$ is a shortest path from s_2 to 1 in $R \backslash H$, and it has length $n - 2$. Thus $\mathrm{diam}(\mathrm{GT}_H(R \backslash H)) \geq n - 2$. □

The following is an example of a ring R such that $\mathrm{GT}_H(R)$ is connected, but $\mathrm{GT}_H(R \backslash H)$ is not connected.

Example 3.6. (a) Let $R = \mathbb{Q}[X]$. Then $H = \mathbb{Q}[X] \backslash \mathbb{Q}^*$ is a multiplicative-prime subset of R that is not an ideal of R. Thus $\mathrm{GT}_H(H)$ is connected with $\mathrm{diam}(\mathrm{GT}_H(H)) = 2$ by Theorem 3.1(1). Moreover, $\mathrm{GT}_H(R)$ is connected with $\mathrm{diam}(\mathrm{GT}_H(R)) = 2$ (by Theorems 3.2 and 3.4) since $R = (X, X + 1)$ with $X, (X + 1) \in H$. However, $\mathrm{GT}_H(R \backslash H)$ is not connected since there is no path from 1 to 2 in $\mathrm{GT}_H(R \backslash H)$. Thus the converse of Theorem 3.1(3) need not hold.

(b) Let $R = \mathbb{Z}$. Then $H = \mathbb{Z}\backslash U(\mathbb{Z})$ is a multiplicative-prime subset of R that is not an ideal of R. Since $\mathrm{GT}_H(R\backslash H)$ is clearly connected, $\mathrm{GT}_H(R)$ is connected by Theorem 3.1(3).

If H is not an ideal of R, then $\mathrm{diam}(\mathrm{GT}_H(H)) = 2$ by Theorem 3.1. Moreover, we have $2 \leq \mathrm{diam}(\mathrm{GT}_H(R)) < \infty$ when $\mathrm{GT}_H(R)$ is connected. In the next example, for every integer $n \geq 2$, we construct a commutative ring R_n such that $\mathrm{GT}_{H_n}(R_n)$ is connected with $\mathrm{diam}(\mathrm{GT}_{H_n}(R_n)) = n$ for some multiplicative-prime subset H_n of R_n.

Example 3.7. (a) Let $n \geq 2$ be an integer, $R_n = \mathbb{Z}[X_1, \ldots, X_{n-1}]$, $P_0 = (X_1 + X_2 + \cdots + X_{n-1})$, $P_i = (X_i)$ for every integer i with $1 \leq i \leq n-2$, and $P_{n-1} = (X_{n-1} + 1)$. Then $P_0, P_1, \ldots, P_{n-1}$ are distinct prime ideals of R_n, and thus $H_n = P_0 \cup P_1 \cup \cdots \cup P_{n-1}$ is a multiplicative-prime subset of R_n. Moreover, $1 = -(X_1 + X_2 + \cdots + X_{n-1}) + X_1 + X_2 + \cdots + X_{n-2} + (X_{n-1} + 1)$ is the sum of n elements of H_n; and by construction, n is the least integer $m \geq 2$ such that R_n is generated by m elements of H_n. Hence $\mathrm{GT}_{H_n}(R_n)$ is connected with $\mathrm{diam}(\mathrm{GT}_{H_n}(R_n)) = n$ by Theorem 3.2 and Theorem 3.4, respectively.

(b) Let $R = \mathbb{Z}[\{X_n\}_{n=1}^{\infty}]$. For a fixed $n \geq 2$, let $P_0, P_1, \ldots, P_{n-1}$, and H_n be defined as in part (a) above (but as ideals in R). Then $\mathrm{GT}_{H_n}(R)$ is connected with $\mathrm{diam}(\mathrm{GT}_{H_n}(R)) = n$ by Theorem 3.2 and Theorem 3.4, respectively.

(c) The Krull dimension hypotheses are needed in the "in particular" statements in Theorems 3.2 and 3.4. Let $R = \mathbb{Z}[X]$, be a two-dimensional integral domain, and $H = (2) \cup (X)$, be a multiplicative-prime subset of R. Then $(H) \subsetneq R$; so $\mathrm{GT}_H(R)$ is not connected by Theorem 3.2. Next, let $I = (2) \cap (X)$, $R' = R/I$, and $H' = (2)/I \cup (X)/I$. Then R' is one-dimensional, H' is a multiplicative-prime subset of R', and $(H') \subsetneq R'$. Thus $\mathrm{GT}_{H'}(R')$ is not connected by Theorem 3.2.

Example 3.6(a) shows that we may have $\mathrm{diam}(\mathrm{GT}_H(R)) < \infty$ and $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = \infty$. The next example shows that we may also have either $\mathrm{diam}(\mathrm{GT}_H(R)) = \mathrm{diam}(\mathrm{GT}_H(R\backslash H))$ or $\mathrm{diam}(\mathrm{GT}_H(R)) > \mathrm{diam}(\mathrm{GT}_H(R\backslash H))$ when H is not an ideal of R.

Example 3.8. (a) Let $R = \mathbb{Z}$ and $H = 3\mathbb{Z} \cup 5\mathbb{Z}$. Then $\mathrm{diam}(\mathrm{GT}_H(R)) = 2$ by Theorem 3.4. Let $x, y \in R\backslash H$ such that $x + y \notin H$ (for example, let $x = 2$, $y = 14$). By the Chinese Remainder Theorem, there is a $w \in R$ such that $w + x, w + y \in H$ (for $x = 2, y = 14$, let $w = 1$). Note that $w \notin H$ since $x, y \notin H$. Thus $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = 2$, and hence $\mathrm{diam}(\mathrm{GT}_H(R)) = \mathrm{diam}(\mathrm{GT}_H(R\backslash H))$.

(b) Let $R = \mathbb{Z}$ and $H = 2\mathbb{Z} \cup 5\mathbb{Z}$. Then $\mathrm{diam}(\mathrm{GT}_H(R)) = 2$ by Theorem 3.4. Since every element of $R\backslash H$ is an odd integer, $x + y \in 2\mathbb{Z} \subseteq H$ for every $x, y \in R\backslash H$. Thus $\mathrm{diam}(\mathrm{GT}_H(R\backslash H)) = 1$, and hence $\mathrm{diam}(\mathrm{GT}_H(R)) > \mathrm{diam}(\mathrm{GT}_H(R\backslash H))$.

In view of Theorems 3.2 and 3.4, we have the following two results. Recall that two ideals I and J of R are *co-maximal* if $R = I + J$. Note that if a multiplicative-prime subset H of R contains two comaximal ideals, then H is not an ideal of R.

Theorem 3.9. *Let R be a commutative ring and H be a multiplicative-prime subset of R that contains two co-maximal ideals of R. Then $\mathrm{GT}_H(R)$ is connected with $\mathrm{diam}(\mathrm{GT}_H(R)) = 2$. In particular, this holds if H is not an ideal of R and either $\dim(R) = 0$ or R is an integral domain with $\dim(R) = 1$.*

Proof. Let $I, J \subseteq H$ be co-maximal ideals of R. Then $R = I + J$; so $R = (i, j)$ for some $i \in I$ and $j \in J$. Thus $\mathrm{GT}_H(R)$ is connected with $\mathrm{diam}(\mathrm{GT}_H(R)) = 2$ by Theorem 3.2 and Theorem 3.4, respectively. The "in particular" statement is clear. □

Corollary 3.10. *Let R be a commutative ring, H be a multiplicative-prime subset of R that is not an ideal of R, $S = R\backslash H$, and R_S be the localization of R with respect to S. Then $\mathrm{GT}_{H_S}(R_S)$ is connected with $\mathrm{diam}(\mathrm{GT}_{H_S}(R_S)) = 2$.*

Proof. Clearly H_S is a multiplicative-prime subset of R_S. Since H is not an ideal of R, there are $x, y \in H$ such that $x + y \in S = R\backslash H$. Since H is a union of prime ideals of R, there are prime ideals P and Q of R contained in H with $x \in P\backslash Q$ and $y \in Q\backslash P$. Thus the prime ideals P_S and Q_S are co-maximal in R_S; so the result follows by Theorem 3.9. □

The following is an example of a commutative ring R with a multiplicative-prime subset H such that neither $\mathrm{GT}_H(R\backslash H)$ nor $\mathrm{GT}_H(R)$ is connected, but $\mathrm{GT}_{H_S}(R_S)$ is connected for some multiplicatively closed subset S of R with $S \neq R\backslash H$.

Example 3.11. Let $R = \mathbb{Z}[X]$, $H = X\mathbb{Z}[X] \cup 3\mathbb{Z}[X]$, and $S = \{1, (X+3), (X+3)^2, (X+3)^3, \ldots\} \subsetneq R\backslash H$. Then H is a multiplicative-prime subset of R that is not an ideal of R and $(H) = (3, X) \subsetneq R$. Thus $\mathrm{GT}_H(R)$ is not connected by Theorem 3.2, and hence $\mathrm{GT}_H(R\backslash H)$ is not connected by Theorem 3.1(3). Since $X\mathbb{Z}[X]_S$, $3\mathbb{Z}[X]_S$ are co-maximal ideals of R_S and $X\mathbb{Z}[X]_S$, $3\mathbb{Z}[X]_S \subseteq H_S$, the graph $\mathrm{GT}_{H_S}(R_S)$ is connected with $\mathrm{diam}(\mathrm{GT}_{H_S}(R_S)) = 2$ by Theorem 3.9.

We next investigate the girth of $\mathrm{GT}_H(H)$, $\mathrm{GT}_H(R\backslash H)$, and $\mathrm{GT}_H(R)$ when H is not an ideal of R. Recall that $|H| \geq 3$ if H is not an ideal of R.

Theorem 3.12. *Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R. Let $H = \bigcup_\alpha P_\alpha$ for prime ideals P_α of R. Suppose that $a - b - c$ is a path of length two in $\mathrm{GT}_H(R\backslash H)$ for distinct vertices $a, b, c \in R\backslash H$.*

(1) *If $2k \in H$ for some $k \in \{a, b, c\}$ and $\bigcap_\alpha P_\alpha \neq \{0\}$, then $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$.*
(2) *If $2k = 0$ for some $k \in \{a, b, c\}$ and $\mathrm{char}(R) \neq 2$, then $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$.*
(3) *If $2k \notin H$ for every $k \in \{a, b, c\}$, then $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) \leq 4$.*

Proof. (1) Suppose that $2k \in H$ for some $k \in \{a, b, c\}$ and there is a $0 \neq h \in \bigcap_\alpha P_\alpha$. Assume $2a \in H$. If $b \neq a+h$, then $a - b - (a+h) - a$ is a cycle of length three in $\mathrm{GT}_H(R\backslash H)$. Hence, assume that $b = a+h$. Since $(a+h)+c = b+c \in H$ and $h \in \bigcap_\alpha P_\alpha$, we have $a+c \in H$. Thus $a - b - c - a$ is a cycle of length three in $\mathrm{GT}_H(R\backslash H)$. Assume $2b \in H$. If $c \neq b+h$, then $b - c - (b+h) - b$ is a cycle of length three in $\mathrm{GT}_H(R\backslash H)$. Thus, assume $c = b+h$. Hence $a - b - (b+h) - a$ is a cycle of length three in $\mathrm{GT}_H(R\backslash H)$. Assume $2c \in H$. If $b \neq c+h$, then $b - c - (c+h) - b$ is a cycle of length three in $\mathrm{GT}_H(R\backslash H)$. Thus, assume that $b = c+h$. Since $a+(c+h) = a+b \in H$ and $h \in \bigcap_\alpha P_\alpha$, we have $a+c \in H$. Hence $a-b-c-a$ is a cycle of length three in $\mathrm{GT}_H(R\backslash H)$. Thus $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$.

(2) Suppose that $2k = 0$ for some $k \in \{a, b, c\}$ and $\mathrm{char}(R) \neq 2$. Thus $2 \neq 0$. Since $k \in R\backslash H$ and $2k = 0$, we have $2 \in P_\alpha$ for every P_α. Hence $0 \neq 2 \in \bigcap_\alpha P_\alpha$. Thus $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$ by part (1) above.

(3) Suppose that $2k \notin H$ for every $k \in \{a, b, c\}$. Then $z \neq -z$ for every $z \in \{a, b, c\}$. Hence there are distinct $x, y \in \{a, b, c\}$ such that $y \neq -x$. Thus $x - y - (-y) - (-x) - x$ is a 4-cycle in $\mathrm{GT}_H(R\backslash H)$; so $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) \leq 4$. □

We will need the following lemma.

Lemma 3.13. *Let R be a commutative ring. Then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if R has two distinct prime ideals P and Q with $|P| = |Q| = 2$.*

Proof. We need only show that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ when R has two distinct prime ideals P and Q with $|P| = |Q| = 2$. Let $P = \{0, x\}$ and $Q = \{0, y\}$ be distinct prime ideals of R, where $x \neq y$ and $x, y \in R^*$. Since $x \notin Q$, we have $x^2 \neq 0$, and thus $x^2 = x$. Since $x(1-x) = 0 \in Q$ and $x \notin Q$, we have $1 - x \in Q$, and hence $1 - x = y \in Q$. Since x and $1 - x = y$ are nonzero idempotent elements of R, we have $R \cong P \times Q$. Thus $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ since P and Q are finite commutative rings with $|P| = |Q| = 2$. □

Theorem 3.14. *Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R.*

(1) *Either* $\mathrm{gr}(\mathrm{GT}_H(H)) = 3$ *or* $\mathrm{gr}(\mathrm{GT}_H(H)) = \infty$. *Moreover, if* $\mathrm{gr}(\mathrm{GT}_H(H)) = \infty$, *then* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $H = Z(R)$; *so* $\mathrm{GT}_H(H)$ *is a* $K^{1,2}$ *star graph with center* 0.
(2) $\mathrm{gr}(\mathrm{GT}_H(R)) = 3$ *if and only if* $\mathrm{gr}(\mathrm{GT}_H(H)) = 3$.
(3) $\mathrm{gr}(\mathrm{GT}_H(R)) = 4$ *if and only if* $\mathrm{gr}(\mathrm{GT}_H(H)) = \infty$ (*if and only if* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$).
(4) *If* $\mathrm{char}(R) = 2$, *then* $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$ *or* ∞. *In particular*, $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$ *if* $\mathrm{char}(R) = 2$ *and* $\mathrm{GT}_H(R\backslash H)$ *contains a cycle.*
(5) $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3, 4,$ *or* ∞. *In particular*, $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) \leq 4$ *if* $\mathrm{GT}_H(R\backslash H)$ *contains a cycle.*

Proof. (1) If $x + y \in H$ for distinct $x, y \in H^*$, then $0 - x - y - 0$ is a 3-cycle in $\mathrm{GT}_H(H)$; so $\mathrm{gr}(\mathrm{GT}_H(H)) = 3$. Otherwise, $x + y \in R\backslash H$ for all distinct

$x, y \in H^*$. So in this case, every $x \in H^*$ is adjacent to 0, and no two distinct $x, y \in H^*$ are adjacent. Thus $\mathrm{GT}_H(H)$ is a star graph with center 0; so $\mathrm{gr}(\mathrm{GT}_H(H)) = \infty$.

Since H is a multiplicative-prime subset of R, we have $H = \bigcup_{\alpha\in\Lambda} P_\alpha$ for distinct prime ideals P_α of R. Also, $|\Lambda| \geq 2$ since H is not an ideal of R. Assume that $\mathrm{gr}(\mathrm{GT}_H(H)) = \infty$. Then $x + y \in R\backslash H$ for all distinct $x, y \in H^*$, and thus every $|P_\alpha| = 2$. Hence the intersection of any two distinct P_α's is $\{0\}$, and thus $|\Lambda| = 2$. (If $P_1, P_2, P_3 \subseteq H$ are distinct prime ideals of R, then $P_1P_2 \subseteq P_1 \cap P_2 = \{0\} \subseteq P_3$. Thus $P_1 \subseteq P_3$ or $P_2 \subseteq P_3$, a contradiction since every $|P_i| = 2$.) Hence $H = P_1 \cup P_2$ for prime ideals P_1, P_2 of R with $P_1 \cap P_2 = \{0\}$ and $|P_1| = |P_2| = 2$, and thus $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by Lemma 3.13. Hence P_1 and P_2 are the only prime ideals of R and $Z(R) = P_1 \cup P_2 = H$.

(2) We need only show that $\mathrm{gr}(\mathrm{GT}_H(H)) = 3$ when $\mathrm{gr}(\mathrm{GT}_H(R)) = 3$. Since $\mathrm{gr}(\mathrm{GT}_H(R)) = 3$ and H is not a prime ideal of R, we have $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ by part (1) above. Thus R has at most one prime ideal, say P, with $|P| = 2$ by Lemma 3.13. Hence, since H is not an ideal of R and H is a union of prime ideals of R, there must be a prime ideal $Q \subsetneq H$ with $|Q| \geq 3$; so $\mathrm{gr}(\mathrm{GT}_H(H)) = 3$.

(3) Suppose that $\mathrm{gr}(\mathrm{GT}_H(H)) = \infty$. Then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H = Z(R)$ by part (1) above; so $\mathrm{gr}(\mathrm{GT}_H(R)) = 4$. Conversely, suppose that $\mathrm{gr}(\mathrm{GT}_H(R)) = 4$. Then $\mathrm{gr}(\mathrm{GT}_H(H)) = \infty$ by parts (1) and (2) above.

(4) Suppose that $\mathrm{char}(R) = 2$ and $\mathrm{GT}_H(R\backslash H)$ contains a cycle C. Then C contains a path $a - b - c$ for some distinct vertices $a, b, c \in R\backslash H$. Since $b \neq c$, we have $0 \neq b + c \in H$. Let $H = \bigcup_\alpha P_\alpha$ for prime ideals P_α of R. Suppose there is a $0 \neq h \in \bigcap_\alpha P_\alpha$. Then $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$ by Theorem 3.12(1). Suppose that $\bigcap_\alpha P_\alpha = \{0\}$. Since $\mathrm{Nil}(R) \subseteq \bigcap_\alpha P_\alpha = \{0\}$, the ring R is reduced. Hence $b^2 + c^2 = (b + c)^2 \neq 0$; so $b^2 \neq c^2$. Thus $b^2 \neq bc$ and $c^2 \neq bc$. For if $b^2 = bc$, then $b(b + c) = 0$, and hence $b + c \in P_\alpha$ for every P_α since $b \notin P_\alpha$ for every P_α. Thus $0 \neq b + c \in \bigcap_\alpha P_\alpha$, a contradiction. Similarly, we cannot have $c^2 = bc$. Since $b, c \in R\backslash H$, we have $bc \notin H$. Also, $b^2 + bc, bc + c^2, c^2 + b^2 \in H$ since $b + c \in H$. Hence $b^2 - bc - c^2 - b^2$ is a 3-cycle in $\mathrm{GT}_H(R\backslash H)$, and thus $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$.

(5) By part (4) above, we may assume that $\mathrm{char}(R) \neq 2$. Suppose that $\mathrm{GT}_H(R\backslash H)$ contains a cycle C. Then C contains a path $a - b - c$, where a, b, c are distinct vertices of $R\backslash H$. Suppose that $2k = 0$ for some $k \in \{a, b, c\}$. Then $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 3$ by Theorem 3.12(2). Thus, assume that $2k \neq 0$ for every $k \in \{a, b, c\}$. Then $z \neq -z$ for every $z \in \{a, b, c\}$, and hence $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) \leq 4$ as in the proof of Theorem 3.12(3). $\square$

The next example shows that the three possibilities for $\mathrm{gr}(\mathrm{GT}_H(R\backslash H))$ when H is not an ideal of R from Theorem 3.14(5) may occur when $\mathrm{gr}(\mathrm{GT}_H(H)) = \mathrm{gr}(\mathrm{GT}_H(R)) = 3$. However, if $\mathrm{gr}(\mathrm{GT}_H(H)) = \infty$ and H is not an ideal of R, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by Theorem 3.14(1), and thus $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = \infty$ and $\mathrm{gr}(\mathrm{GT}_H(R)) = 4$. In particular, $\mathrm{gr}(\mathrm{GT}_H(H)) = 3$ when R is not reduced and H is not an ideal of R.

Example 3.15. (a) Let $R = \mathbb{Z}$ and H be the union of all the prime ideals of R; so $R\backslash H = U(\mathbb{Z}) = \{1, -1\}$. It is easy to check that $\mathrm{gr}(\mathrm{GT}_H(H)) = \mathrm{gr}(\mathrm{GT}_H(R)) = 3$ and $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = \infty$.

(b) Let $R = \mathbb{Z}$ and $H = 2\mathbb{Z} \cup 3\mathbb{Z}$. Then it is easy to check that $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = \mathrm{gr}(\mathrm{GT}_H(H)) = \mathrm{gr}(\mathrm{GT}_H(R)) = 3$.

(c) Let $R = (\mathbb{Z}_3 \times \mathbb{F}_4)[X]$ and H be the union of all the prime ideals of R. Then $R\backslash H = U(R) = U(\mathbb{Z}_3) \times U(\mathbb{F}_4) = (\mathbb{Z}_3)^* \times (\mathbb{F}_4)^*$. It is easy to check that $\mathrm{GT}_H(R\backslash H) = K^{3,3}$. Thus $\mathrm{GT}_H(R\backslash H)$ (and hence $\mathrm{GT}_H(R)$) is connected and $\mathrm{gr}(\mathrm{GT}_H(R\backslash H)) = 4$. It is also easy to check that $\mathrm{gr}(\mathrm{GT}_H(R)) = \mathrm{gr}(\mathrm{GT}_H(H)) = 3$.

4. $\mathbf{GT}_{\boldsymbol{H}}(\boldsymbol{R})$ for Specific Rings

In this section, we determine $\mathrm{GT}_H(R)$ for several classes of commutative rings. In particular, we investigate $\mathrm{GT}_H(R)$ when R is either an idealization, a $D + M$ construction, or a localization. We start with some examples.

Example 4.1. (a) Let R be a commutative ring such that its set of prime ideals is totally ordered under inclusion (e.g. R is a valuation domain). Then every multiplicative-prime subset H of R is a prime ideal of R; so the structure of $\mathrm{GT}_H(R)$ and $\mathrm{GT}_H(R\backslash H)$ is completely described in Sec. 2.

(b) Let R be a commutative ring with $\dim(R) = 0$ (e.g. R is finite). If R is quasilocal with maximal ideal M, then $H = M = Z(R)$; so $\mathrm{GT}_H(R) = T(\Gamma(R))$. So we may assume that R is not quasilocal. If H is an (prime) ideal of R, then the structure of $\mathrm{GT}_H(R)$ and $\mathrm{GT}_H(R\backslash H)$ is described in Sec. 2. Otherwise, H contains two comaximal ideals; so $\mathrm{GT}_H(R)$ is connected with $\mathrm{diam}(\mathrm{GT}_H(R)) = 2$ by Theorem 3.9.

(c) Let R be a quasilocal commutative ring with maximal ideal M and H be a multiplicative-prime subset of R. If H is an (prime) ideal of R, then the structure of $\mathrm{GT}_H(R)$ and $\mathrm{GT}_H(R\backslash H)$ is described in Sec. 2. If H is not an (prime) ideal of R, then $\mathrm{GT}_H(R)$ is not connected by Corollary 3.3 since $(H) \subseteq M \subsetneq R$.

(d) Let R be an integral domain with $\dim(R) = 1$ and H a multiplicative-prime subset of R. If H is an (prime) ideal of R, then the structure of $\mathrm{GT}_H(R)$ and $\mathrm{GT}_H(R\backslash H)$ is described in Sec. 2. Otherwise, H contains two comaximal ideals; so $\mathrm{GT}_H(R)$ is connected with $\mathrm{diam}(\mathrm{GT}_H(R)) = 2$ by Theorem 3.9.

The next result follows directly from the definitions.

Theorem 4.2. *Let H_1 and H_2 be multiplicative-prime subsets of a commutative ring R. Then $\mathrm{GT}_{H_1}(R) \subseteq \mathrm{GT}_{H_2}(R)$ if and only if $H_1 \subseteq H_2$. In particular, $\mathrm{GT}_{H_1}(R) = \mathrm{GT}_{H_2}(R)$ if and only if $H_1 = H_2$. Moreover, if $R = Z(R) \cup U(R)$ (e.g. $\dim(R) = 0$ or R is finite), then $\mathrm{GT}_H(R)$ is a subgraph of $T(\Gamma(R))$.*

Theorem 4.3. *Let $R \subseteq T$ be an extension of commutative rings, and let H be a multiplicative-prime subset of T. Then $H' = H \cap R$ is a multiplicative-prime subset of R and* $\mathrm{GT}_{H'}(R) \subseteq \mathrm{GT}_H(T)$. *Moreover,* $\mathrm{GT}_H(T)$ *is connected if* $\mathrm{GT}_{H'}(R)$ *is connected, and* $\mathrm{diam}(\mathrm{GT}_H(T)) \leq \mathrm{diam}(\mathrm{GT}_{H'}(R))$.

Proof. The first part of the theorem is clear. For the "moreover" part, note that $(H) = T$ if $(H') = R$. Thus $\mathrm{GT}_H(T)$ is connected if $\mathrm{GT}_{H'}(R)$ is connected by Corollary 3.3 and $\mathrm{diam}(\mathrm{GT}_H(T)) \leq \mathrm{diam}(\mathrm{GT}_{H'}(R))$ by Theorem 3.4. □

However, $\mathrm{GT}_H(T)$ may be connected when $\mathrm{GT}_{H'}(R)$ is not connected. Let $R = \mathbb{Z} \subseteq T = \mathbb{Z}[X]$ and $H = (X) \cup (X+1)$. Then $\mathrm{GT}_H(T)$ is connected with $\mathrm{diam}(\mathrm{GT}_H(T)) = 2$ by Theorem 3.9. But $H' = H \cap R = \{0\}$ is a prime ideal of R; so $\mathrm{GT}_{H'}(R)$ is not connected by Theorem 2.1.

Recall that for an R-module M, the *idealization of M over R* is the commutative ring formed from $R \times M$ by defining addition and multiplication as $(r,m)+(s,n) = (r+s, m+n)$ and $(r,m)(s,n) = (rs, rn+sm)$, respectively. A standard notation for this "idealized ring" is $R(+)M$; see [3, 18] for basic properties of rings resulting from the idealization construction. The zero-divisor graph $\Gamma(R(+)M)$ has been studied in [10, 13], and the total graph $T(\Gamma(R(+)M))$ has been studied in [23].

Let M be an R-module. Since $(\{0\}(+)M)^2 = 0$, it is easy to check that F is a multiplicative-prime subset of $T = R(+)M$ if and only if $F = H(+)M$, where H is a multiplicative-prime subset of R. Moreover, F is an (prime) ideal of T if and only if H is an (prime) ideal of R; and if H is an ideal of R, then $T/F \cong R/H$. The next theorem thus follows directly from Theorems 2.3 and 2.5.

Theorem 4.4. *Let R be a commutative ring, H be a prime ideal of R, M be a nonzero R-module, $T = R(+)M$, and $F = H(+)M$.*

(1) $\mathrm{GT}_F(T \backslash F)$ *is complete if and only if* $R/H \cong \mathbb{Z}_2$.
(2) $\mathrm{GT}_F(T \backslash F)$ *is connected if and only if* $\mathrm{GT}_H(R \backslash H)$ *is connected, if and only if either* $R/H \cong \mathbb{Z}_2$ *or* $R/H \cong \mathbb{Z}_3$.
(3) (a) $\mathrm{diam}(\mathrm{GT}_F(T \backslash F)) = 1$ *if and only if* $R/H \cong \mathbb{Z}_2$.
    (b) $\mathrm{diam}(\mathrm{GT}_F(T \backslash F)) = 2$ *if and only if* $R/H \cong \mathbb{Z}_3$.
    (c) *Otherwise,* $\mathrm{diam}(\mathrm{GT}_F(T \backslash F)) = \infty$.

We next consider the case when H is not an ideal of R.

Theorem 4.5. *Let R be a commutative ring, H be a multiplicative-prime subset of R that is not an ideal of R, M be an R-module, $T = R(+)M$, and $F = H(+)M$. Then* $\mathrm{GT}_F(T)$ *is connected if and only if* $\mathrm{GT}_H(R)$ *is connected, and moreover,* $\mathrm{diam}(\mathrm{GT}_F(T)) = \mathrm{diam}(\mathrm{GT}_H(R))$.

Proof. It is easily verified that $(H) = R$ if and only if $(F) = T$ since $(h_1, \ldots, h_n) = R$ for $h_i \in H$ if and only if $((h_1, m_1), \ldots, (h_n, m_n)) = T$ for $(h_i, m_i) \in T$. Thus

$\mathrm{GT}_F(T)$ is connected if and only if $\mathrm{GT}_H(R)$ is connected by Theorem 3.2 and $\mathrm{diam}(\mathrm{GT}_F(T)) = \mathrm{diam}(\mathrm{GT}_H(R))$ by Theorem 3.4. □

Next, we consider the $D+M$ construction as in [17]. Let T be an integral domain of the form $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Then for D a subring of K, $R = D + M$ is a subring of T with the same quotient field as T. This construction has proved very useful for constructing examples. If P is a prime ideal of D, then $Q = P + M$ is a prime ideal of R with $R/Q \cong D/P$. Since any multiplicative-prime subset is a union of prime ideals, it follows that $F = H + M$ is a multiplicative-prime subset of R for $H \subseteq D$ if and only if H is a multiplicative-prime subset of D. Note that if H is an (prime) ideal of D, then F is an (prime) ideal of R with $R/F \cong D/H$. Thus Theorems 2.3 and 2.5 yield an analog of Theorem 4.4 for $\mathrm{GT}_F(R)$; we leave the details to the interested reader.

Theorem 4.6. *Let $T = K+M$ be an integral domain, where K is a subfield of T and M is a nonzero maximal ideal of T, D be a subring of K, and $R = D+M$. Let H be a multiplicative-prime subset of D and $F = H+M$. Then $\mathrm{GT}_F(R)$ is connected if and only if $\mathrm{GT}_H(D)$ is connected, and moreover,* $\mathrm{diam}(\mathrm{GT}_F(R)) = \mathrm{diam}(\mathrm{GT}_H(D))$.

Proof. It is easily verified that $(H) = D$ if and only if $(F) = R$. Thus D is connected if and only if R is connected by Corollary 3.3. Moreover, if $(h_1, \ldots, h_n)D = D$ for $h_i \in H$, then $(h_1, \ldots, h_n)R = R$. Conversely, if $(h_1 + m_1, \ldots, h_n + m_n) = R$ for $h_i + m_i \in F$, then $(h_1, \ldots, h_n)D = D$. Hence $\mathrm{diam}(\mathrm{GT}_F(R)) = \mathrm{diam}(\mathrm{GT}_H(D))$ by Theorem 3.4. □

Let R be an integral domain and S be a multiplicatively closed subset of R; so $R \subseteq R_S$. Let H be a multiplicative-prime subset of R with $H \cap S = \emptyset$. Then H is a union of prime ideals of R disjoint from S; so H_S is a union of prime ideals of R_S. Thus H_S is a multiplicative-prime subset of R_S. If H is a prime ideal of R, then H_S is a prime ideal of R_S with $R_S/H_S \cong R/H$. Thus Theorems 2.3 and 2.5 yield an analog of Theorem 4.4 for $\mathrm{GT}_{H_S}(R_S)$; we again leave the details to the interested reader.

Theorem 4.7. *Let S be a multiplicatively closed subset of an integral domain R and H be a multiplicative-prime subset of R with $H \cap S = \emptyset$. Then $\mathrm{GT}_{H_S}(R_S)$ is connected if and only if $\mathrm{GT}_H(R)$ is connected, and moreover,* $\mathrm{diam}(\mathrm{GT}_{H_S}(R_S)) = \mathrm{diam}(\mathrm{GT}_H(R))$.

Proof. It is easily verified that $(H) = R$ if and only if $(H_S) = R_S$. Thus $\mathrm{GT}_{H_S}(R_S))$ is connected if and only if $\mathrm{GT}_H(R)$ is connected by Corollary 3.3. Moreover, $(h_1, \ldots, h_n) = R$ for $h_i \in H$ if and only if $(h_1/s_1, \ldots, h_n/s_n) = R_S$ for $h_i \in H, s_i \in S$. Hence $\mathrm{diam}(\mathrm{GT}_{H_S}(R_S)) = \mathrm{diam}(\mathrm{GT}_H(R))$ by Theorem 3.4. □

In [6, Corollary 3.6], we showed that if R has a nontrivial idempotent, then $T(\Gamma(R))$ is connected with $\mathrm{diam}(T(\Gamma(R)) = 2$. In particular, this holds for a (nontrivial) product of commutative rings (see [6, Corollary 3.7]). The following example shows that this need not hold for $\mathrm{GT}_H(R)$. However, note that if $\dim(R) = 0$, then $\mathrm{GT}_H(R)$ is connected with $\mathrm{diam}(\mathrm{GT}_H(R)) = 2$ when H is not an ideal of R by Theorem 3.2 and Theorem 3.4, respectively.

Example 4.8. Let $R = \mathbb{Z}[X] \times \mathbb{Z}$ and $H = (X\mathbb{Z}[X] \times \mathbb{Z}) \cup (3\mathbb{Z}[X] \times \mathbb{Z})$. Then R has nontrivial idempotents, H is a multiplicative-prime subset of R, and $(H) \neq R$. Thus $\mathrm{GT}_H(R)$ is not connected by Theorem 3.2.

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